

Non-linear Analysis of 2D Frame with Semi-Rigid Connections

1. Outline of a program

- This is a program for Geometrically Nonlinear Analysis of 2D-Frame, and large displacement of elastic frame members can be simulated.
- Incremental Arc-Length Method is used to solve the incremental stiffness equations.
- Internal forces of the frame members are calculated using the linear stiffness matrix and the displacements without rigid body displacement.
- Simultaneous equations which has been linearized by incremental method are solved using Gauss-Jordan elimination. Because the dimension of matrix is not so large and the stiffness matrix sometimes becomes negative-definite.
- Input/Output file name can be defined arbitrarily with the format of 'csv' and those are inputted from command line of a terminal.
- Used language for program is 'Fortran 90' and used compiler is 'GNU gfortran.'
- Please note that it is assumed the directions of external loads don't change before and after deformation.

2. Incremental stiffness equation

2.1 For Normal Beam Element

As well known, a tangential stiffness matrix including a geometrical nonlinear terms can be expressed for normal beam element shown below:

$$\{\Delta f\} = [k_T]\{\Delta u\} \quad [k_T] = [k_L] + [k_G]$$

$$\{\Delta f\} = \{\Delta N_i \quad \Delta S_i \quad \Delta M_i \quad \Delta N_j \quad \Delta S_j \quad \Delta M_j\}^T$$

$$\{\Delta u\} = \{\Delta u_i \quad \Delta v_i \quad \Delta \theta_i \quad \Delta u_j \quad \Delta v_j \quad \Delta \theta_j\}^T$$

$$[k_L] = \begin{bmatrix} EA/\ell & 0 & 0 & -EA/\ell & 0 & 0 \\ 0 & 12EI/\ell^3 & 6EI/\ell^2 & 0 & -12EI/\ell^3 & 6EI/\ell^2 \\ 0 & 6EI/\ell^2 & 4EI/\ell & 0 & -6EI/\ell^2 & 2EI/\ell \\ -EA/\ell & 0 & 0 & EA/\ell & 0 & 0 \\ 0 & -12EI/\ell^3 & -6EI/\ell^2 & 0 & 12EI/\ell^3 & -6EI/\ell^2 \\ 0 & 6EI/\ell^2 & 2EI/\ell & 0 & -6EI/\ell^2 & 4EI/\ell \end{bmatrix}$$

$$[K_G] = P \cdot \begin{bmatrix} 1/\ell & 0 & 0 & -1/\ell & 0 & 0 \\ 0 & 6/5\ell & 1/10 & 0 & -6/5\ell & 1/10 \\ 0 & 1/10 & 2\ell/15 & 0 & -1/10 & -\ell/30 \\ -1/\ell & 0 & 0 & 1/\ell & 0 & 0 \\ 0 & -6/5\ell & -1/10 & 0 & 6/5\ell & -1/10 \\ 0 & 1/10 & -\ell/30 & 0 & -1/10 & 2\ell/15 \end{bmatrix}$$

- $[k_T]$: Tangential stiffness matrix of an element
- $[k_L]$: Linear stiffness matrix of an element
- $[k_G]$: Geometrical non-linear stiffness matrix of an element
- P : Axial force of an element (positive sign means tension)
- $\{\Delta f\}$: Incremental equivalent nodal force vector of an element
- $\{\Delta u\}$: Incremental displacement vector of an element
- E, A, I, ℓ : Elastic modulus, section area, moment of inertia, element length
- $\Delta N, \Delta S, \Delta M$: Increments of axial force, shear force and bending moment
- $\Delta u, \Delta v, \Delta \theta$: Increments of displacements in axial direction, transverse direction and rotation
- Subscripts i and j : variables for node i and node j

2.2 For Beam Element with Semi-Rigid Connections

When a beam element with semi-rigid connection wants to be used, a tangential stiffness matrix of an element becomes followings. This stiffness matrix is introduced in the paper 'Optimization of Frames with Semi-Rigid Connections, L.M.C.Simoës, Computers & Structures Vol.60, No.4, pp.531-539, 1996.'

$$[k_T] = [k_S] + [k_G]$$

$$[k_S] = \begin{bmatrix} EA/\ell & 0 & 0 & -EA/\ell & 0 & 0 \\ 0 & 12EI/\ell^3 \cdot (A/H) & 6EI/\ell^2 \cdot (B/H) & 0 & -12EI/\ell^3 \cdot (A/H) & 6EI/\ell^2 \cdot (G/H) \\ 0 & 6EI/\ell^2 \cdot (B/H) & 4EI/\ell \cdot (C/H) & 0 & -6EI/\ell^2 \cdot (B/H) & 2EI/\ell \cdot (F/H) \\ -EA/\ell & 0 & 0 & EA/\ell & 0 & 0 \\ 0 & -12EI/\ell^3 \cdot (A/H) & -6EI/\ell^2 \cdot (B/H) & 0 & 12EI/\ell^3 \cdot (A/H) & -6EI/\ell^2 \cdot (G/H) \\ 0 & 6EI/\ell^2 \cdot (G/H) & 2EI/\ell \cdot (F/H) & 0 & -6EI/\ell^2 \cdot (G/H) & 4EI/\ell \cdot (D/H) \end{bmatrix}$$

$$g_1 = \frac{1}{1 + 3EI/S_i L} \quad g_2 = \frac{1}{1 + 3EI/S_j L}$$

$$\begin{aligned} A &= g_1 + g_2 + g_1 \cdot g_2 & B &= g_1 \cdot (2 + g_2) & C &= 3 \cdot g_1 \\ D &= 3 \cdot g_2 & F &= 3 \cdot g_1 g_2 & G &= g_2 \cdot (2 + g_1) \\ H &= 4 - g_1 \cdot g_2 \end{aligned}$$

where, S_i and S_j are rotational spring stiffness at the two ends of a beam element.

When $S_i, S_j \rightarrow \infty$, $g_1 = g_2 = 1$ and $A = B = C = D = F = G = H = 3$, therefore $[k_S] = [k_L]$.

3. Calculation formulas for Internal forces

3.1 Calculation of internal force

The increments of internal force $\{\Delta r\}$ can be obtained using $[k_L]$ or $[k_S]$ and $\{\Delta u^*\}$. Where, $[k_L]$ is a linear stiffness matrix of an element, $[k_S]$ is a stiffness matrix of an element with semi-rigid connections. and $\{\Delta u^*\}$ is an incremental displacement vector ecluding rigid rotation in local coordinate system.

$$\{\Delta r\} = [k_L]\{\Delta u^*\} \quad \text{or} \quad \{\Delta r\} = [k_S]\{\Delta u^*\}$$

$$\begin{aligned} \{\Delta r\} &= \{\Delta N_i^* \quad \Delta S_i^* \quad \Delta M_i^* \quad \Delta N_j^* \quad \Delta S_j^* \quad \Delta M_j^*\}^T \\ \{\Delta u^*\} &= \{\Delta u_i^* \quad \Delta v_i^* \quad \Delta \theta_i^* \quad \Delta u_j^* \quad \Delta v_j^* \quad \Delta \theta_j^*\}^T \end{aligned}$$

$$\begin{aligned} \Delta u_i^* &= 0 & \Delta v_i^* &= 0 & \Delta \theta_i^* &= (\tan \theta_i^*)_k - (\tan \theta_i^*)_{k-1} \\ \Delta u_j^* &= \Delta \ell & \Delta v_j^* &= 0 & \Delta \theta_j^* &= (\tan \theta_j^*)_k - (\tan \theta_j^*)_{k-1} \end{aligned}$$

Where, $\Delta \ell$ means the difference of the previous element length and current element length. Regarding the rotation component, it is taken as the difference of the previous rotation angle (subscript $k-1$) and current rotation angle (subscript k).

3.2 Elimination of rigid rotation

The method of elimination of rigid rotation is shown below:

Using the addition formula for tangent,

$$\begin{aligned} \frac{dv^*}{dx^*}|_i &= \tan \theta_i^* = \tan(\theta_i - R) = \frac{\tan \theta_i - \tan R}{1 + \tan \theta_i \tan R} = \frac{(\ell + u_j - u_i) \tan \theta_i - (v_j - v_i)}{(\ell + u_j - u_i) + (v_j - v_i) \tan \theta_i} \\ \frac{dv^*}{dx^*}|_j &= \tan \theta_j^* = \tan(\theta_j - R) = \frac{\tan \theta_j - \tan R}{1 + \tan \theta_j \tan R} = \frac{(\ell + u_j - u_i) \tan \theta_j - (v_j - v_i)}{(\ell + u_j - u_i) + (v_j - v_i) \tan \theta_j} \\ \left(\tan R &= \frac{v_j - v_i}{\ell + u_j - u_i}, \quad \frac{dv}{dx}|_i = \tan \theta_i, \quad \frac{dv}{dx}|_j = \tan \theta_j \right) \end{aligned}$$

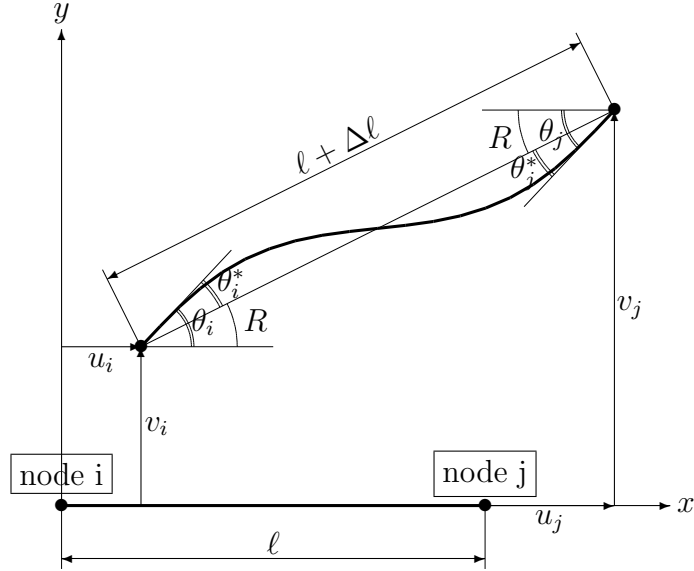


Fig.1 Elimination of Rigid Body Rotation

4. Arc-Length Method

4.1 Initial value of $\Delta\lambda$

When simplified scalar load-displacement curve is considered, following equations can be obtained referring below figure.

$$\begin{aligned} K_T \cdot \Delta U &= \Delta\lambda \cdot \Delta F + \Delta R \\ \Delta U &= \Delta\lambda \cdot \Delta U_0 + \Delta U_R \end{aligned}$$

$$\begin{aligned} (\Delta s)^2 &= (\Delta U)^2 + (\Delta\lambda \cdot \phi \Delta F)^2 \\ &= (\Delta\lambda)^2 \{(\Delta U_0)^2 + (\phi \Delta F)^2\} + 2\lambda \cdot (\Delta U_0 \cdot \Delta U_R) + (\Delta U_R)^2 \end{aligned}$$

$$\Delta\lambda = \frac{-(\Delta U_0 \cdot \Delta U_R) \pm \sqrt{\{(\Delta U_0)^2 + (\phi \Delta F)^2\} \cdot (\Delta s)^2 - (\phi \Delta F \cdot \Delta U_R)^2}}{(\Delta U_0)^2 + (\phi \Delta F)^2}$$

where, K_T : Tangential stiffness $\Delta\lambda$: Coefficient for external force
 ΔF : External force increment ΔU_0 : Displacement increment for external force
 ΔR : Unbalanced force increment ΔU_R : Displacement increment for unbalanced
 ΔU : Displacement increment Δs : Arc length
 ϕ : Scaling parameter

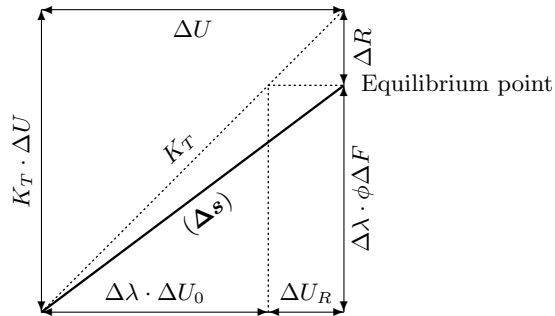


Fig.2 Concept of Arc-Length Method

If $\Delta U_R = 0$ is assumed, initial value of $\Delta\lambda$ can be obtained as following equation.

$$\Delta\lambda_0 = \pm \sqrt{\frac{(\Delta s)^2}{(\Delta U_0)^2 + (\phi\Delta F)^2}}$$

Above equation can take two values, and it should be noted that a sign of $\Delta\lambda_0$ is very important in Arc-Length method.

A sign of $\Delta\lambda_0$ is defined as shown below:

- Define a displacement increment vector $\{\Delta U_{-1}\}$ from previous equilibrium point to current equilibrium point.
- Calculate $\{\Delta U_0\}$ as $\{\Delta U_0\} = [K_T]^{-1}\{\Delta F\}$.
- Calculate an inner product $\{\Delta U_{-1}\}^T\{\Delta U_0\} = |\Delta U_{-1}| \cdot |\Delta U_0| \cdot \cos\theta$, where θ is an angle between 2 displacement increment vectors.
- If an inner product $\{\Delta U_{-1}\}^T\{\Delta U_0\} \geq 0$, the angle θ is less than or equal to 90 degree. In this case, $\Delta\lambda_0$ has positive sign.
- If an inner product $\{\Delta U_{-1}\}^T\{\Delta U_0\} < 0$, the angle θ is greater than 90 degree. In this case, $\Delta\lambda_0$ has negative sign.

Regarding the scaling parameter ϕ , it can be obtained as following equation. In this program, recommended value of α is one ($\alpha = 1.0$).

$$\phi = \sqrt{\frac{\alpha}{\{\Delta F\}^T\{\Delta F\}}}$$

Regarding the arc length Δs , it can be obtained assuming $\Delta\lambda = 1.0$.

$$\Delta s = \sqrt{\{\Delta U_0\}^T\{\Delta U_0\} + \phi^2 \cdot \{\Delta F\}^T\{\Delta F\}} \quad (\Delta\lambda_0 = 1.0)$$

4.2 Correction factor $\Delta\lambda$ for iterative calculation

Referring above conceptual figure and replacing Δs to ΔL , following equation can be obtained.

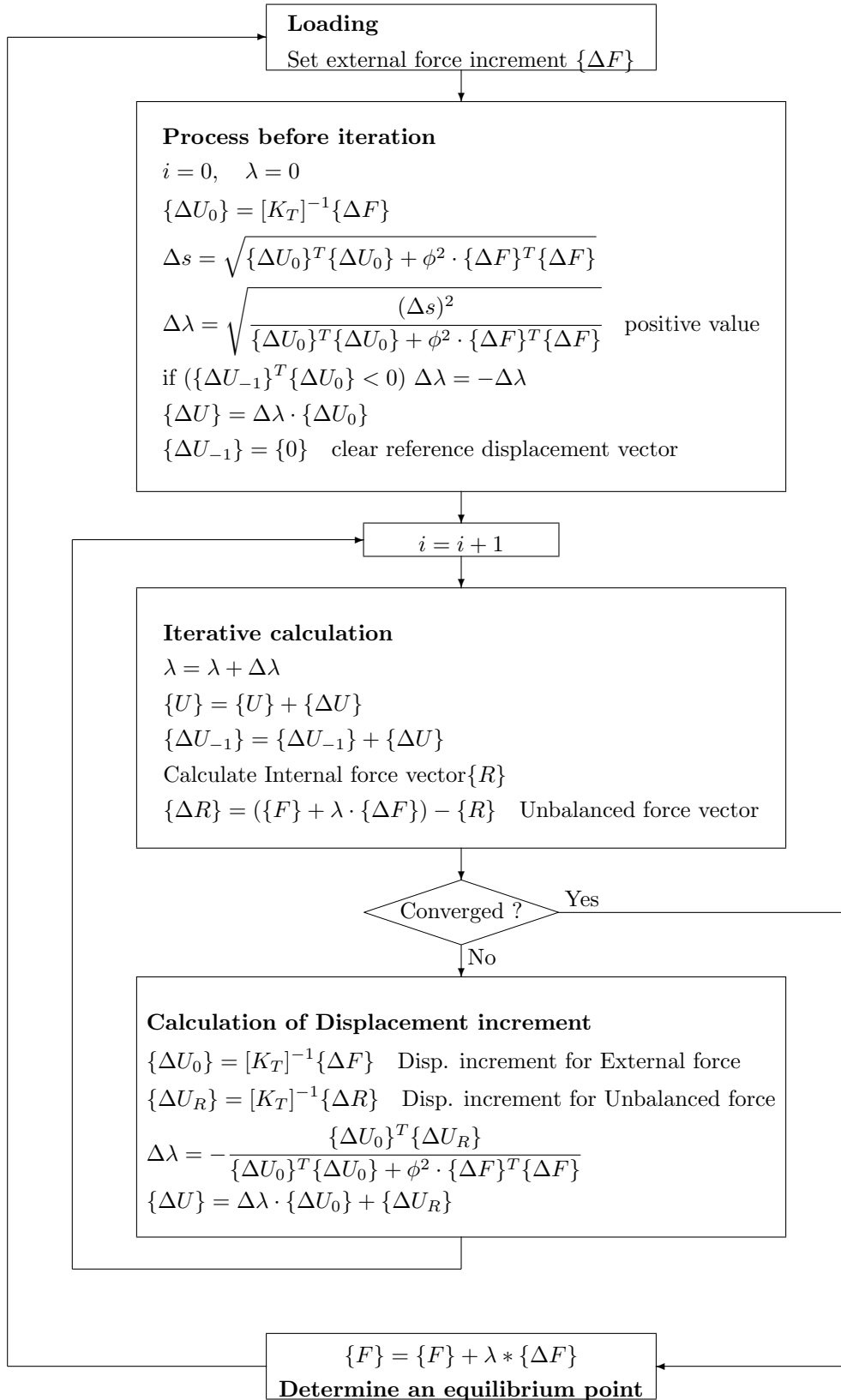
$$\begin{aligned} (\Delta L)^2 &= (\Delta U)^2 + (\Delta\lambda \cdot \phi\Delta F)^2 \\ &= (\Delta\lambda)^2\{(\Delta U_0)^2 + (\phi\Delta F)^2\} + 2\Delta\lambda \cdot (\Delta U_0 \cdot \Delta U_R) + (\Delta U_R)^2 \end{aligned}$$

From the condition of minimization of $(\Delta L)^2$, $\Delta\lambda$ can be calculate as following equation.

$$\frac{d(\Delta L)^2}{d\Delta\lambda} = 0 \quad \rightarrow \quad \Delta\lambda = -\frac{\Delta U_0 \cdot \Delta U_R}{(\Delta U_0)^2 + (\phi\Delta F)^2}$$

4.3 Flowchart for analysis

- $\{F\}$: Total external force vector at equilibrium point
- $\{\Delta F\}$: External incremental force vector
- $\{R\}$: Internal force vector
- $\{\Delta R\}$: Unbalanced force vector
- $\{U\}$: Total displacement vector
- $\{\Delta U\}$: Displacement increment vector
- $\{\Delta U_0\}$: Displacement increment vector for external force increment
- $\{\Delta U_R\}$: Displacement increment vector for unbalanced force
- $\{\Delta U_{-1}\}$: Displacement increment vector from previous equilibrium point to current equilibrium point
- $[K_T]$: Tangential stiffness matrix including non-linear component
- Δs : Arc length
- λ : Coefficient for external force increment and displacement increment
- $\Delta\lambda$: Increment of coefficient λ
- ϕ : Scaling parameter



note) Simultaneous equations are solved using Gauss-Jordan elimination.
Condition for convergence

$$\frac{\sum |\Delta U|}{\sum |U|} < 1e-6 \quad \frac{\sum |\Delta R|}{\sum |R|} < 1e-3$$

Fig.3 Flowchart for Arc-Length Method

5. Example of numerical calculation

5.1 subject

To calculate critical loads of multi-story one bay frame with semi-rigid connections. Finally, the relationship between critical load and fixity factor which is defined as follow:

$$\gamma = \frac{1}{1 + 3EI/SL}$$

where, S is a rotational spring stiffness of semi-rigid connection.

5.2 Model

Members properties and calculation models are shown below. For each model, initial geometrical imperfection of 2mm at top right column is considered, because to estimate buckling loads using nonlinear step analysis.

Table1 Members properties

Member	Vertical	Horizontal
Type of element	Normal beam	Semi-rigid connection
Elastic modulus	210 GPa	210 GPa
Section area	33.4 cm ²	43.0 cm ²
Moment of inertia	1510 cm ⁴	2770 cm ⁴

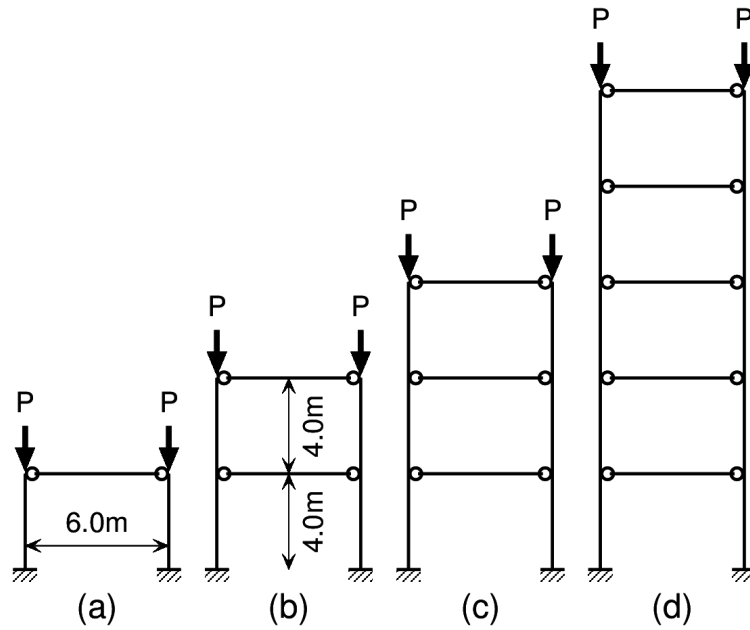


Fig.4 Calculation Model

6. Calculation Results

Fig.5 shows Load-Displacement curves and Load- Δs curves for each model. The displacement u and v are horizontal displacement and vertical displacement respectively at right-top node.

When load- Δs curves are compared with load-displacement curve, it can be understood that although Δs has constant value under small load, Δs becomes large as a bifurcation point approaches. Considering above phenomenon, a load at changing point of Δs has been deemed as a bifurcation point which is equal to critical load.

It is easy to know this method gives smaller critical load than actual critical load. However, this method is convenience to estimate approximate critical load.

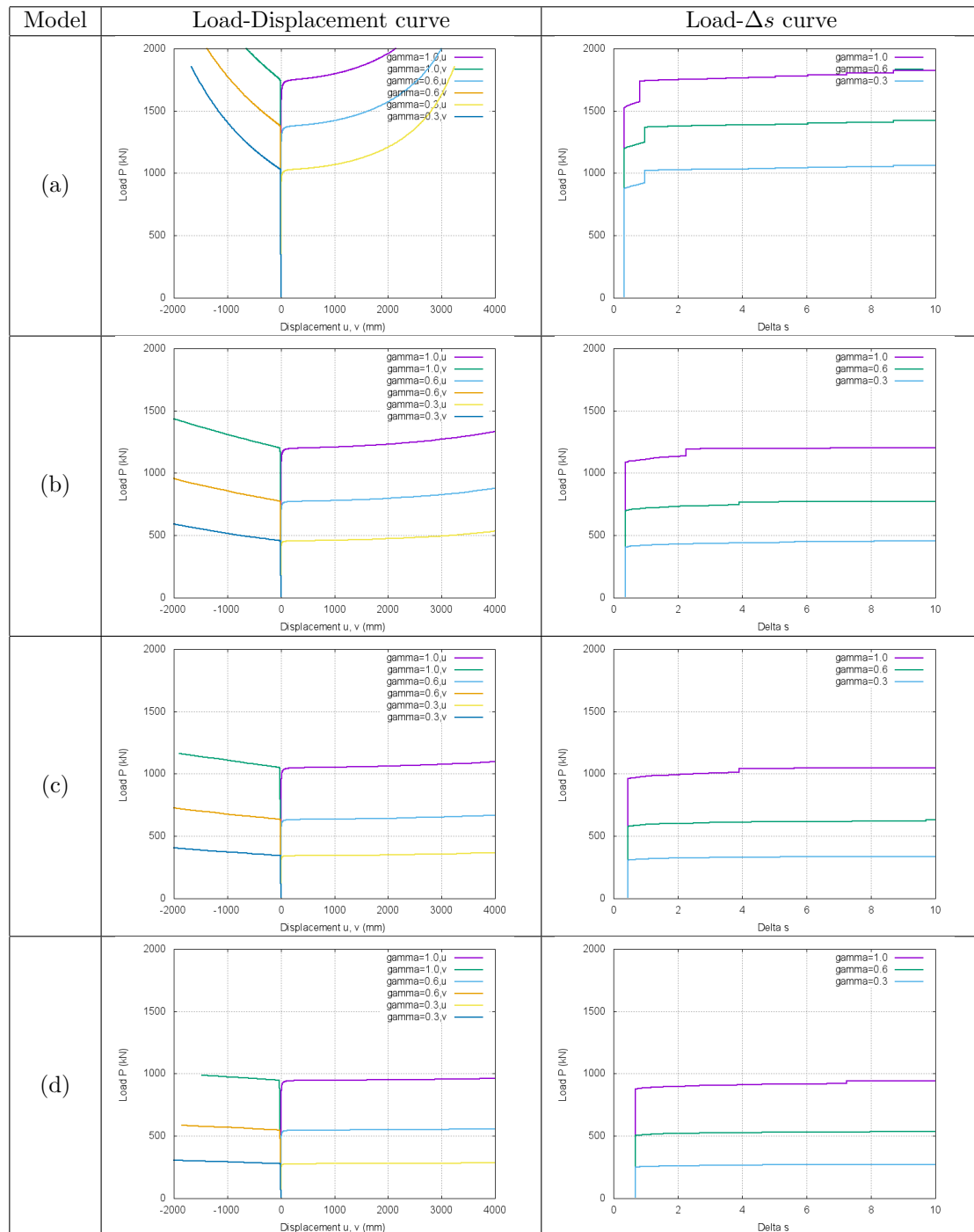


Fig.5 Calculation Results

Table2 Calculation results of Critical loads

γ	Critical load P (kN)				Normarized Critical load			
	(a)	(b)	(c)	(d)	(a)	(b)	(c)	(d)
0.0	483.5	107.1	44.5	15.8	0.315	0.098	0.046	0.018
0.1	627.2	208.6	131.0	92.1	0.409	0.192	0.136	0.104
0.2	756.5	309.2	219.5	171.4	0.493	0.284	0.228	0.195
0.3	881.4	408.3	308.4	252.1	0.575	0.375	0.320	0.286
0.4	997.6	507.3	397.4	334.9	0.651	0.466	0.412	0.380
0.5	1105.4	604.9	487.9	419.0	0.721	0.555	0.506	0.475
0.6	1205.2	701.0	580.3	506.9	0.786	0.644	0.602	0.575
0.7	1297.3	799.3	673.1	596.2	0.846	0.734	0.698	0.677
0.8	1381.7	896.1	767.9	688.3	0.901	0.823	0.797	0.781
0.9	1458.4	993.6	864.8	783.3	0.951	0.912	0.897	0.889
1.0	1533.5	1089.0	963.9	881.3	1.000	1.000	1.000	1.000

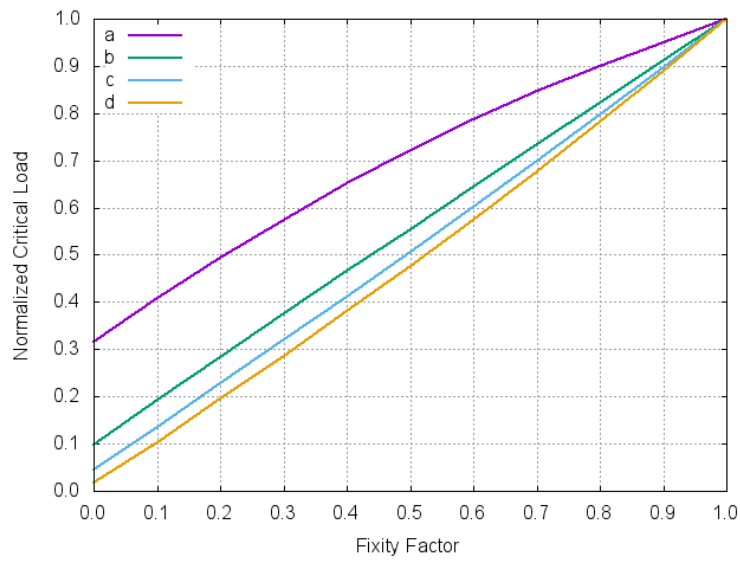


Fig.6 Influence of Connection Flexibility on the Critital Load